MATH 1010E University Mathematics
Lecture Notes (week 11)
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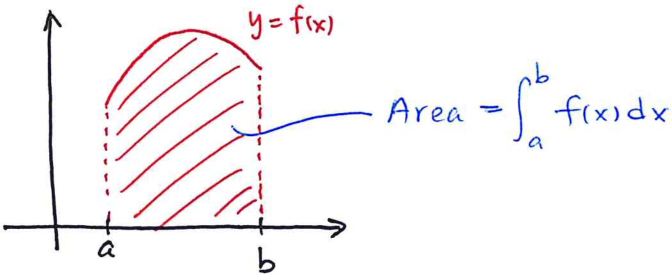
1 Indefinite vs definite integrals

So far we have learned two kinds of integrals: *indefinite* and *definite*. Given a function $f(x)$, the indefinite integral concerns finding a primitive function $F(x)$ such that $F'(x) = f(x)$, we write

$$\int f(x)\, dx = F(x) + C.$$

On the other hand , the definite integral concerns finding the area under the graph of $y = f(x)$ above some closed and bounded interval $[a, b]$.



If f is continuous on $[a.b]$, then we can calculate the definite integral using the Riemann sum

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\Delta x_k.$$

Even though these two kinds of integrals are defined very differently, they are indeed closely related to each other through the *Fundamental Theorem of Calculus*: which says that if $F'(x) = f(x)$, then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

In other words, definite integrals can be computed by first finding the primitive function $F(x)$ using indefinite integrals and then doing a substitution. Therefore, if we can find the indefinite integral, then the definite integral can be easily found. So the main question is the following:

Question: Given $f(x)$, when can we "find" the primitive function $F(x)$?

There are two meanings of "finding" a primitive function: whether it exists at all and whether we can write it down in terms of elementary functions. The second part is more complicated since it is not universal to say which functions are "elementary". For example, the indefinite integral

$$\int x^2 \sin x^x \, dx$$

is very hard to find, but we can ask if a primitive function $F(x)$ should exist in the first place, without knowing a formula for $F(x)$. This is answered by one part of the Fundamental Theorem of Calculus.

2 Fundamental Theorem of Calculus I

The first part of fundamental theorem of calculus basically says that continuity is all that is required to guarantee the existence of a primitive function.

Theorem 2.1 (Fundamental Theorem of Calculus I) *If $f : [a, b] \to \mathbb{R}$ is a continuous function, then the function $F : [a, b] \to \mathbb{R}$ given by*

$$F(x) := \int_a^x f(t) \, dt$$

is well-defined, differentiable in (a, b) and satisfies

$$F'(x) = f(x) \qquad \text{for all } x \in (a, b),$$

i.e. $F(x)$ is a primitive function of $f(x)$.

Remark 2.2 *The theorem does not say anything if f is not continuous. In particular, it does not mean that a primitive function would not exists if the function f is not continuous. On the other hand, the theorem only tells us about the existence, but not how to find a formula for $F(x)$ (note that it is defined in terms of a definite integral).*

Before we discuss the proof of Theorem 2.1, we need to study more properties of definite integrals (since $F(x)$ is defined as such).

3 Preliminary results on definite integrals

The main result in this section is the integral mean value theorem (Theorem 3.X), which is similar in spirit to the differentiable mean value theorem which says that

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \qquad \text{for some } \xi \in (a, b)$$

if f is differentiable on (a, b) and continuous on $[a, b]$.

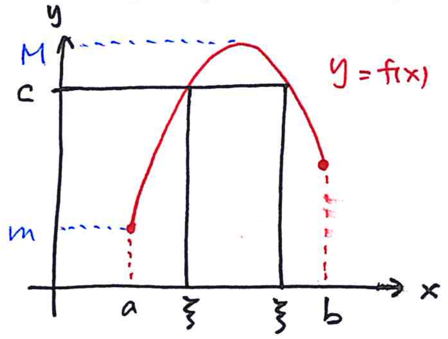
We need the following theorem about continuous functions, which says that continuous functions cannot "jump".

Theorem 3.1 *If $f : [a, b] \to \mathbb{R}$ is a continuous function, let*

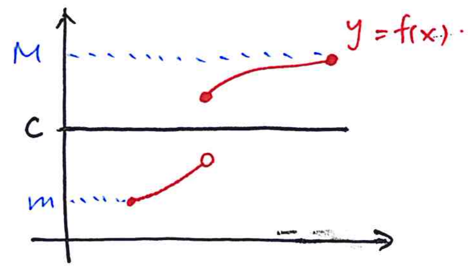
$$m := \min_{x \in [a,b]} f(x) \qquad \text{and} \qquad M := \max_{x \in [a,b]} f(x),$$

then for any $c \in [m, M]$, there exists $\xi \in [a, b]$ such that $f(\xi) = c$.

Note that ξ is not necessarily unique. See for the picture below:



Note that the continuity assumption is important. If the function f is continuous, the theorem fails even if the maximum and minimum are defined. For example,



The next result says that definite integrals preserve the ordering.

Theorem 3.2 *If $g, f, h : [a, b] \to \mathbb{R}$ are continuous functions such that*

$$g(x) \le f(x) \le h(x) \qquad \text{for all } x \in [a, b],$$

then

$$\int_a^b g(x)\, dx \le \int_a^b f(x)\, dx \le \int_a^b h(x)\, dx.$$

Question: Try to give a proof of Theorem 3.2 above. Note that you can reduce it to the case that if $f(x) \geq 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.

Question: Do we have a similar statement for differentiation? That is, does $f(x) \geq 0$ implies $f'(x) \geq 0$?

We now state the main result in this section.

Theorem 3.3 (Integral Mean Value Theorem) *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then*

$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

for some $\xi \in [a,b]$.

Proof: Since f is continuous on a closed and bounded interval $[a,b]$, by the Extreme Value Theorem,

$$m := \min_{x\in[a,b]} f(x) \qquad \text{and} \qquad M := \max_{x\in[a,b]} f(x) \quad \text{exist.}$$

Therefore,

$$m \leq f(x) \leq M \qquad \text{for all } x \in [a,b].$$

By Theorem 3.2, we have

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a).$$

Dividing by $b-a$ we get

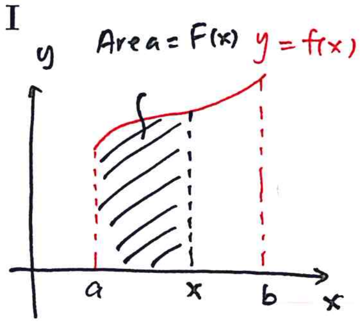
$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

By Intermediate Value Theorem, there exists $\xi \in [a,b]$ such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This proves the theorem after multiplying $(b-a)$ to the left hand side.

4 Proof of Fundamental Theorem of Calculus I



Given f continuous on $[a,b]$, the function

$$F(x) := \int_a^x f(t)\,dt$$

is well-defined since f is also continuous on each subinterval $[a,x]$.

Claim: $F'(x) = f(x)$ for all $x \in (a,b)$.

To prove the claim, we use the definition of derivative. Let $h > 0$.

$$\begin{aligned}
\frac{F(x+h)-F(x)}{h} &= \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right) \\
&= \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right) \\
&= \frac{1}{h}\int_x^{x+h} f(t)\,dt \\
&= \frac{1}{h}(f(\xi)\cdot h) \\
&= f(\xi)
\end{aligned}$$

for some $\xi \in [x, x+h]$. The case is similar for $h < 0$. Therefore, we have proved that, by taking $h \to 0$,

$$F'(x) = \lim_{\xi \to x} f(\xi) = f(x),$$

where the last equality holds since f is continuous.

Remark 4.1 *We can rephrase the Fundamental Theorem of Calculus I as*

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

In other words, if we integrate a function f first and then differentiate, we get back the original function.

What about if we differentiate first and then integrate? This is answered by the second part of Fundamental Theorem of Calculus.

5 Fundamental Theorem of Calculus II

Theorem 5.1 (Fundamental Theorem of Calculus II) *Let $F : [a, b] \to \mathbb{R}$ be a differentiable function such that $F'(x)$ is continuous on $[a, b]$. Then,*

$$\int_a^b F'(x)\, dx = F(b) - F(a).$$

Remark 5.2 *We can also rephrase the above formula as*

$$\int_a^b \frac{d}{dx} F(x)\, dx = F(b) - F(a).$$

Therefore, differentiating first and then integrate gives us the original function "in some sense". The two Fundamental Theorems thus say that differentiation *and* integration *are reverse processes of each other.*

Proof: As in Fundamental Theorem of Calculus I, we define the function $G : [a, b] \to \mathbb{R}$ by

$$G(x) := \int_a^x f(t)\, dt$$

and we know that $G'(x) = f(x)$ for all $x \in (a, b)$. If we let $H : [a, b] \to \mathbb{R}$ be

$$H(x) := F(x) - G(x).$$

Note that

$$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \qquad \text{for all } x \in (a, b).$$

Therefore, we know that H is a constant function (see assignment 2). In particular, we must have $H(a) = H(b)$. On the other hand,

$$H(a) = F(a) - G(a) = F(a) - \int_a^a f(t)\, dt = F(a) - 0 = F(a),$$

$$H(b) = F(b) - G(b) = F(b) - \int_a^b f(x)\, dx.$$

Therefore, combining these, we have

$$F(a) = F(b) - \int_a^b f(x)\, dx.$$

Rearranging gives

$$\int_a^b F'(x)\, dx = \int_a^b f(x)\, dx = F(b) - F(a).$$

6 Integration by parts

From the Fundamental Theorem of Calculus II, calculating definite integrals amounts to first finding the indefinite integral $F(x)$ and then doing substitution. So we focus on developing more techniques to find indefinite integrals. We have already learned how to integrate some elementary functions like x^n, $\sin x$ or e^x. We have also learned the method of u-substitution which is a consequence of chain rule in differentiation.

There is another useful integration technique called *integration by part*, which is a consequence of the product rule in differentiation. Recall that the product rule says that if u and v are two differentiable functions, then

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}.$$

We can express it in "*differential form*" as

$$d(uv) = vdu + udv.$$

If we integrate on both sides and apply the fundamental theorem, we get

$$uv = \int d(uv) = \int v\,du + \int u\,dv.$$

Therefore, we have

$$\int v\,du = uv - \int u\,dv.$$

In other words, we can switch u and v with an induced "$-$" sign and also a new product term uv. This is called *integration by part*. Let us look at a few examples.

Example 6.1 *Consider the integral*

$$\int \ln x\,dx.$$

If we let $v = \ln x$ and $u = x$, then integration by part gives

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int xd(\ln x)\\ &= x\ln x - \int x\cdot\frac{1}{x}\,dx\\ &= x\ln x - x + C.\end{aligned}$$

One can easily check the answer by differentiating

$$(x\ln x + x)' = x\cdot\frac{1}{x} + \ln x - 1 = \ln x.$$

Example 6.2 *Consider the integral*

$$\int x^2 e^{-2x}\,dx.$$

Note that $d(-\frac{1}{2}e^{-2x}) = e^{-2x}dx$, therefore

$$\begin{aligned}
\int x^2 e^{-2x}\,dx &= \int x^2 d(-\frac{1}{2}e^{-2x}) \\
&= -\frac{1}{2}x^2e^{-2x} + \frac{1}{2}\int e^{-2x}d(x^2) \\
&= -\frac{1}{2}x^2e^{-2x} + \int xe^{-2x}\,dx \\
&= -\frac{1}{2}x^2e^{-2x} + \int x\,d(-\frac{1}{2}e^{-2x}) \\
&= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx \\
&= -\frac{1}{2}x^2e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C \\
&= -\frac{1}{4}e^{-2x}(2x^2 + 2x + 1) + C.
\end{aligned}$$

Example 6.3 *Consider the integral*

$$\int x\tan^{-1} x\,dx.$$

Sometimes we have to choose how to apply integration by part. The idea is that we want the integral to simplify as much as possible. Recall that

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \text{and} \qquad \frac{d}{dx}(x\tan^{-1} x) = \tan^{-1} x + \frac{x}{1+x^2}.$$

Let's try both ways to se which one is simpler. Using the first observation.

$$\begin{aligned}
\int x\tan^{-1} x\,dx &= \int \tan^{-1} x\, d(\frac{x^2}{2}) \\
&= \frac{x^2}{2}\tan^{-1} x - \int \frac{x^2}{2}\,d(\tan^{-1} x) \\
&= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx \\
&= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)\,dx \\
&= \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C.
\end{aligned}$$

Using the second approach.

$$\begin{aligned}\int x\tan^{-1}x\,dx &= x^2\tan^{-1}x - \int x\,d(x\tan^{-1}x)\\ &= x^2\tan^{-1}x - \int x\tan^{-1}x + \frac{x^2}{1+x^2}\,dx\\ &= x^2\tan^{-1}x - (x-\tan^{-1}x) - \int x\tan^{-1}x\,dx\end{aligned}$$

Putting the integral back to the left hand side, we get

$$2\int x\tan^{-1}x\,dx = x^2\tan^{-1}x - (x-\tan^{-1}x) + C,$$

which gives us the same answer after dividing by 2.

Example 6.4 *Consider the integral*

$$\int \sin^{-1}x\,dx.$$

$$\begin{aligned}\int \sin^{-1}x\,d &= x\sin^{-1}x - \int x\,d(\sin^{-1}x)\\ &= x\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx\\ &= x\sin^{-1}x + \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}}\\ &= x\sin^{-1}x + \sqrt{1-x^2} + C.\end{aligned}$$

7 A review of trigonometric identities

When dealing with integrals of trigonometric functions, it is often useful to apply some trigonometric identities to simplify or transform the integrals. Let us review some trigonometric identities in this section.

Proposition 7.1 (Basic identities) *The following identities hold:*

(a) $\cos^2 x + \sin^2 x = 1.$

(b) $1 + \tan^2 x = \sec^2 x.$

(c) $1 + \cot^2 x = \csc^2 x$.

Note that (b) and (c) follow easily from (a).

Proposition 7.2 (Sum-to-product Formula) *The following identities hold:*

(a) $\cos(x + y) = \cos x \cos y - \sin x \sin y$.

(b) $\sin(x + y) = \sin x \cos y + \cos x \sin y$.

(c) $\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$.

We can prove (a) and (b) easily using Euler's formula:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

Recall that $i^2 = -1$. From the multiplicative property of exponential function,

$$e^{i(x+y)} = e^{ix} e^{iy}.$$

Using Euler's formula,

$$\cos(x + y) + i \sin(x + y) = (\cos x + i \sin x)(\cos y + i \sin y).$$

Expanding the right hand side and compare coefficients, we obtain identities (a) and (b).

Exercise: Derive (c) from (a) and (b).

Setting $x = y$ in the sum-to-product formula, we obtain the following.

Proposition 7.3 (Double angle formula) *The following identities hold:*

(a) $\cos 2x = \cos^2 x - \sin^2 x = 1 - 2\sin^2 x = 2\cos^2 x - 1$.

(b) $\sin 2x = 2 \sin x \cos x$.

(c) $\tan 2x = \dfrac{2 \tan x}{1 - \tan^2 x}$.

A direct consequence of (a) is

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \text{and} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

This is useful since the power is reduced!

Proposition 7.4 (Product-to-sum formula) *The following identities hold:*

(a) $\cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y))$.

(b) $\cos x \sin y = \frac{1}{2}(\sin(x+y) - \sin(x-y))$.

(c) $\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y))$.

Exercise: Use the sum-to-product formula to prove Proposition 7.4 above.

8 Trigonometric integrals

Example 8.1 *Consider the integral*

$$\int \sin^4 x \, dx$$

We can use the double angle formula twice to do this.

$$\begin{aligned}
\int \sin^4 x \, dx &= \int (\sin^2 x)^2 \, dx \\
&= \int \left(\frac{1-\cos 2x}{2}\right)^2 dx \\
&= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx \\
&= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) dx \\
&= \frac{1}{4}\int (\frac{3}{2} - 2\cos 2x + \frac{\cos 4x}{2}) \, dx \\
&= \frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{\sin 4x}{8}\right) + C.
\end{aligned}$$

Question: Evaluate $\int \sin^{2n} x \, dx$ and $\int \cos^{2n} x \, dx$.

Example 8.2 *Consider the integral*

$$\int \sin 3x \sin 5x \, dx.$$

Using the product-to-sum formula, we get

$$\begin{aligned}\int \sin 3x \sin 5x \, dx &= \int \frac{1}{2}(\cos(-2x) - \cos 8x) \, dx \\ &= \frac{1}{2}\int (\cos 2x - \cos 8x) \, dx \\ &= \frac{1}{2}\left(\frac{\sin 2x}{2} - \frac{\sin 8x}{8}\right) + C.\end{aligned}$$

Question: Evaluate $\int \cos 2x \sin 3x \, dx$.

Example 8.3 *Consider the integral*

$$\int \cos x \sin^4 x \, dx.$$

Sometimes we can just do a simple u-substitution to handle trigonometric integrals.

$$\int \cos x \sin^4 x \, dx = \int \sin^4 x \, d(\sin x) = \frac{\sin^5 x}{5} + C.$$

Question: Evaluate $\int \cos^8 x \sin x \, dx$.

Example 8.4 *Consider the integral*

$$\int \sin^2 x \cos^2 x \, dx$$

Using the double angle formula,

$$\begin{aligned}\int \sin^2 x \cos^2 x \, dx &= \int \left(\frac{\sin 2x}{2}\right)^2 dx \\ &= \frac{1}{4}\int \sin^2 2x \, dx \\ &= \frac{1}{4}\int \frac{1 - \cos 4x}{2} \, dx \\ &= \frac{1}{4}\left(\frac{x}{2} - \frac{\sin 4x}{8}\right) + C.\end{aligned}$$

Let us look at two more examples which are a bit trickier.

Example 8.5 *Consider the integral*

$$\int \sec^3 x \, dx.$$

$$\begin{aligned}
\int \sec^3 x \, dx &= \int \sec x \, d(\tan x) \\
&= \sec x \tan x - \int \tan x \, d(\sec x) \\
&= \sec x \tan x - \int \tan^2 x \sec x \, dx \\
&= \sec x \tan x - \int (\sec^3 x - \sec x) \, dx \\
&= \sec x \tan x + \ln|\sec x + \tan x| - \int \sec^3 x \, dx
\end{aligned}$$

This implies that

$$\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C.$$

Example 8.6 *Consider the integral*

$$\int \frac{\sin x}{\cos x + \sin x} \, dx.$$

Using the sum-to-product formula for tan:

$$\begin{aligned}
\int \frac{\sin x}{\cos x + \sin x} \, dx &= \int \frac{\tan x}{1 + \tan x} \, dx \\
&= \frac{1}{2} \int \left(1 - \tan(\frac{\pi}{4} - x)\right) \, dx \\
&= \frac{1}{2}(x + \ln|\sec(\frac{\pi}{4} - x)|) + C.
\end{aligned}$$